ANALYSIS OF THE THERMAL PROCESS IN A RECTANGULAR PARALLELEPIPED AND SOLID CYLINDER FOR A TIME-DEPENDENT TEMPERATURE OF THE AMBIENT MEDIUM

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A method is presented for computing the temperature change based on the exact solution of the heat conduction equation under the assumption of the Newton cooling law. The temperature of the ambient medium is here approximated by a piecewise-linear function of the time.

The heating or cooling of bodies subjected to a time-dependent temperature of the ambient medium occurs quite often in practice. It is interesting to find the temperature distribution for these cases.

If it is assumed that the body is heated (cooled) in a furnace, then the temperature in the combustion chamber will be taken as the temperature of the ambient medium. However, it is impossible to give an analytical expression for the temperature in the combustion chamber of any furnace since a large number of parameters (such as the power used, the specific heat of the heat loss, as well as the state of the regulator, for example) affect the temperature characteristic. Hence, the single reasonable method of giving the time dependence of the temperature in practice is the analytical approximation of empirical data. The approximation of the temperature–time function by means of a piecewise-linear function turns out to be especially simple. Such a form permits achievement of any accuracy, as well as being convenient for mathematical processing.

A curve of furnace heating approximated by the inscribed polygonal line is shown in Fig. 1.

It is easy to see that the heating temperature $\vartheta_m(t)$ for the time interval $t_m \le t \le t_{m+1}$ has the form

$$\vartheta_m(t) = P_m t + \sum_{n=1}^{m-1} (P_n - P_{n+1}) t_{n+1} + Q_1. \tag{1}$$

If any homogeneous and isotropic body with surface A on which natural heat exchange occurs in conformity with the Newton cooling law is considered, then the temperature $\vartheta(P, t)$ at the point P within the body is determined up to the time t by the solution of the boundary-value problem

$$\begin{gathered} D\{\vartheta(P, t)\} = \frac{\partial \vartheta(P, t)}{\partial t} - a\,\Delta\vartheta(P, t) = 0, \ \vartheta(P, 0) = \vartheta_a = \text{const}, \\ L\{\vartheta(P, t)\} = \left[\frac{\lambda}{\alpha} \cdot \frac{\partial \vartheta(P, t)}{\partial n} + \vartheta(P, t)\right]_A = \varphi(t), \ t > 0. \end{gathered} \tag{2}$$

The function $\varphi(t)$ is the time-varying temperature of the ambient medium, and it will henceforth be replaced by the functions $\vartheta_m(t)$ in conformity with (1).

By using the expression

$$\vartheta(P, t) = \vartheta_0(P, t) + \vartheta_1(P, t) \tag{3}$$

we obtain from the system (2) for the function $\vartheta_0(P, t)$

$$\begin{gathered} D\{\vartheta_0(P, t)\} = 0, \ \vartheta_0(P, 0) = \vartheta_a; \\ L\{\vartheta_0(P, t)\} = 0, \ t > 0 \end{gathered} \tag{4}$$

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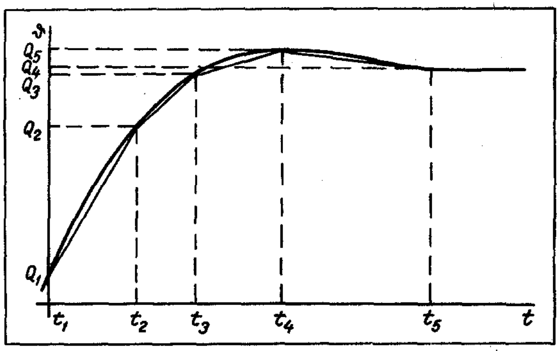

Fig. 1. Furnace heating curve.

and for the function $\vartheta_1(P, t)$

$$D\{\vartheta_1(P, t)\} = 0, \ \vartheta_1(P, 0) = 0; \quad L\{\vartheta_1(P, t)\} = \varphi(t). \tag{5}$$

The solution of the homogeneous problem (4) is standard and has the form

$$\vartheta_0(P, t) = \sum_{\nu=1}^{\infty} A_\nu u_\nu(P) \exp(-a\delta_\nu^2 t), \tag{6}$$

where $u_\nu(P)$ is the complete system of eigenfunctions with corresponding eigenvalues δ_ν. From the initial condition we obtain the relationship

$$\vartheta_a = \sum_{\nu=1}^{\infty} A_\nu u_\nu(P), \tag{7}$$

from which the Fourier coefficients are determined

$$A_\nu = \vartheta_a \int_k u_\nu(P)\, dV. \tag{8}$$

We find the solution of the inhomogeneous problem (5) from [1] by using the Duhamel theorem

$$\vartheta_1(P, t) = \int_0^t \varphi(\lambda) \frac{\partial}{\partial t} \vartheta_2(P, t-\lambda)\, d\lambda, \tag{9}$$

where $\vartheta_2(P, t)$ is the solution of the inhomogeneous problem

$$D\{\vartheta_2(P, t)\} = 0, \ \vartheta_2(P, 0) = 0; \quad L\{\vartheta_2(P, t)\} = 1, \ t > 0. \tag{10}$$

By means of the substitution

$$\vartheta_3(P, t) = \vartheta_2(P, t) - 1 \tag{11}$$

it can be reduced to the homogeneous problem

$$D\{\vartheta_3(P, t)\} = 0, \ \vartheta_3(P, 0) = -1, \quad L\{\vartheta_3(P, t)\} = 0, \ t > 0, \tag{12}$$

whose solution is already known. Using (6)-(8), we obtain

$$\vartheta_2(P, t) = 1 - \sum_{\nu=1}^{\infty} \bar{A}_\nu u_\nu(P) \exp(-a\delta_\nu^2 t), \tag{13}$$

$$1 = \sum_{\nu=1}^{\infty} \bar{A}_\nu u_\nu(P) \tag{14}$$

and

$$\bar{A}_\nu = \int_k u_\nu(P)\, dV. \tag{15}$$

Substituting (9) into (13) we find

$$\vartheta_1(P, t) = \int_0^t \varphi(\lambda) \sum_{\nu=1}^{\infty} \bar{A}_\nu u_\nu(P) \delta_\nu^2 a \exp(-a\delta_\nu^2 (t-\lambda))\, d\lambda = \sum_{\nu=1}^{\infty} \bar{A}_\nu u_\nu(P) \int_0^t \varphi(\lambda)\, a\delta_\nu^2 \exp(-a\delta_\nu^2 (t-\lambda))\, d\lambda. \tag{16}$$

An expression for the ambient temperature varying along the polygonal line is introduced into this equation as the temperature of the ambient medium as a function of the time $\varphi(t)$. Since (1) is valid only for the time interval, it is necessary to separate the integral in (16) also in conformity with the domains of the intervals. We obtain

$$\vartheta_1(P, t) = \sum_{\nu=}^{\infty} \bar{A}_\nu u_\nu(P) \Big[\int_0^{t_2} \vartheta_1(\lambda)\, a\delta_\nu^2 \exp(-a\delta_\nu^2 (t-\lambda))\, d\lambda + \int_{t_2}^{t_3} \vartheta_2(\lambda)\, a\delta_\nu^2 \exp(-a\delta_\nu^2 (t-\lambda))\, d\lambda$$
$$+ \int_{t_m}^{t} \vartheta_m(\lambda)\, a\delta_\nu^2 \exp(-a\delta_\nu^2 (t-\lambda))\, d\lambda \Big]. \tag{17}$$

Using (1), we find for the integrand

$$J_k = \exp(-a\,\delta_\nu^2 t)\int_{t_k}^{t_{k+1}} \vartheta_k(\lambda)\, a\,\delta_\nu^2 \exp(a\,\delta_\nu^2\lambda)\, d\lambda$$

$$= \left[\left(P_k - \frac{P_k}{a\,\delta_\nu^2} + \sum_{n=1}^{k-1}(P_n - P_{n-1})\, t_{n+1} + Q_1\right)\exp(a\delta_\nu^2\lambda)\right]_{\lambda=t_k}^{\lambda=t_{k+1}} \exp(-a\delta_\nu^2 t). \tag{18}$$

If the integral is determined, then we obtain for (17)

$$\vartheta_1(P,\, t) = \sum_{\nu=1}^{\infty}\bar{A}_\nu u_\nu(P)\left\{P_m t + \sum_{n=1}^{m-1}(P_n - P_{n-1})\, t_{n+1} + Q_1\right\}$$

$$-\sum_{\nu=1}^{\infty}\bar{A}_\nu u_\nu(P)\left\{\sum_{n=1}^{m-1}\frac{(P_n - P_{n+1})\exp(a\,\delta_\nu^2(t_{n+1}-t))}{a\,\delta_\nu^2} + \left(Q_1 - \frac{P_1}{a\delta_\nu^2}\right)\exp(-a\delta_\nu^2) + \frac{P_m}{a\delta_\nu^2}\right\} \tag{19}$$

for the domain $t_m \leq t \leq t_{m+1}$.

If (1) and (14) are used for the first expression in the right side, then we obtain

$$\vartheta_1(P,\, t) = \vartheta_m(t) - \sum_{\nu=1}^{\infty}\bar{A}_\nu u_\nu(P)\left\{\sum_{n=1}^{m-1}\frac{(P_n - P_{n+1})\exp(a\,\delta_\nu^2(t_{n+1}-t))}{a\delta_\nu^2}\right.$$

$$\left. + \left(Q_1 - \frac{P_1}{a\delta_\nu^2}\right)\exp(-a\delta_\nu^2 t) + \frac{P_m}{a\delta_\nu^2}\right\} \tag{20}$$

for the time interval $t_m \leq t \leq t_{m+1}$.

We therefore find the general solution for the heat conduction equation for the case when the temperature of the ambient medium varying according to the polygonal line is given on the surface A. In conformity with (3) and the solutions (6) and (20), and also because of the relationship $A_\nu = \vartheta_a \cdot \bar{A}_\nu$ which follows from (8) and (15), we obtain

$$\vartheta(P,\, t) = \vartheta_m(t) - \sum_{\nu=1}^{\infty}\bar{A}_\nu u_\nu(P)\left\{\sum_{n=1}^{m-1}\frac{(P_n - P_{n+1})\exp(a\delta_\nu^2(t_{n+1}-t))}{a\,\delta_\nu^2}\right.$$

$$\left. + \left(Q_1 + \vartheta_a - \frac{P_1}{a\delta_\nu^2}\right)\exp(-a\delta_\nu^2 t) + \frac{P_m}{a\,\delta_\nu^2}\right\} \tag{21}$$

for the time interval $t_m \leq t \leq t_{m+1}$.

Particular solutions for separate body shapes can be obtained from the general solution (21) and are distinguished only by different eigenvalues which can be found in the literature.

The difficulties which can originate in solving multidimensional problems can easily be eliminated since, in conformity with (4) or (12), we obtain the multidimensional solution in the form of the product of one-dimensional solutions for appropriate individual coordinate directions for the homogeneous problem.

Using one-dimensional solutions for plates with diverse heat-transfer coefficients [2] on both boundary surfaces, we obtain the following solution for a rectangular parallelepiped:

$$\vartheta(x_1,\, x_2,\, x_3,\, t) = P_m t + \sum_{n=1}^{m-1}(P_n - P_{n+1})\, t_{n+1} + Q_1 - \sum_{i1=1}^{\infty}\sum_{i2=1}^{\infty}\sum_{i3=1}^{\infty}\bar{A}_{i1}$$

$$\times \bar{A}_{i2}\bar{A}_{i3}\prod_{i_\nu=1}^{i_\nu=3}\left[\frac{k_{i\nu}}{h_{0\nu}}\cos(k_{i\nu}x_\nu) + \sin(k_{i\nu}x_\nu)\right]\left\{\frac{P_m}{aE}\right.$$

$$\left. + \sum_{n=1}^{m-1}\frac{(P_n - P_{n+1})\exp(aE(t_{n+1}-t))}{aE} + \left(Q_1 + \vartheta_a - \frac{P_1}{aE}\right)\exp(-aEt)\right\} \tag{22}$$

for $t_m \leq t$, where l_ν is the length of an edge of the rectangular parallelepiped, $h_{l\nu}$ is the relative heat-transfer coefficient λ/α on the surface $x_\nu = 1$; $h_{0\nu}$ is the relative heat-transfer coefficient λ/α on the surface $x_\nu = 0$;

$$E=(k_{i1}^2+k_{i2}^2+k_{i3}^2); \tag{23}$$

$$\bar{A}_{i\nu}=2\left[1-\cos(k_{i\nu}l_\nu)-\frac{k_{i\nu}}{h_{0\nu}}\sin(k_{i\nu}l_\nu)\right]/k_{i\nu}l_\nu\left\{1+\left(\frac{k_{i\nu}}{h_{0\nu}}\right)^2+\frac{1}{h_{0\nu}l_\nu}+\frac{1}{h_{l\nu}l_\nu}\left[\cos(k_{i\nu}l_\nu)-\frac{k_{i\nu}}{h_{0\nu}}\sin(k_{i\nu}l_\nu)\right]\right\} \tag{24}$$

and $k_{i\nu}$ is given by the eigenvalue equation

$$\cot(k_{i\nu}l_\nu)=\frac{1}{h_{0\nu}+h_{l\nu}}\left\{\frac{k_{i\nu}(\nu)}{l_\nu}-\frac{(h_{0\nu}l_\nu)(h_{l\nu}l_\nu)}{(k_{i\nu}l_\nu)}\right\}. \tag{25}$$

Using the solutions for a plate and an infinite cylinder from [1], we obtain the following solution for a solid infinite cylinder by the same method:

$$\vartheta(r,\ z,\ t)=P_m t+\sum_{n=1}^{m-1}(P_n-P_{n+1})t_{n+1}+Q_1-\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\bar{A}_i\bar{A}_j\left[\frac{k_i}{h_0}\cos(k_i z)+\sin(k_i z)\right]J_0\left(\frac{c_j}{R}r\right)\left\{\frac{P_m}{aE_1}\right.$$
$$\left.+\sum_{n=1}^{m-1}\frac{(P_n-P_{n+1})\exp(aE_1(t_{n+1}-t))}{aE_1}+\left(Q_1+\vartheta_a-\frac{P_1}{aE_1}\right)\exp(-aE_1t)\right\} \tag{26}$$

for $t_m \leq t$, where l is the height of the cylinder, R is the cylinder radius, h_R is the relative heat-transfer coefficient λ/α on the surface r = R, h_0 is the relative heat-transfer coefficient λ/α on the surface z = 0, h_l is the relative heat-transfer coefficient λ/α on the surface z = 1,

$$E_1=(k_i^2+(c_j/R)^2); \tag{27}$$

$$\bar{A}_j=\frac{2}{c_jJ_j(c_j)\left[1+\left(\frac{c_j}{Rh_R}\right)^2\right]}; \tag{28}$$

$\bar{A}_i$ is analogous to (24) and k_i and c_j from the equations for the eigenvalues

$$\cot(k_il)=\frac{1}{h_0+h_l}\left\{\frac{(k_il)}{l}-\frac{(h_0l)(h_ll)}{(k_il)}\right\}, \tag{29}$$

$$\frac{J_0(c_j)}{J_1(c_j)}=\frac{c_j}{Rh_R}. \tag{30}$$

The numerical solution for a solid cylinder can be obtained by using an electronic computer.

LITERATURE CITED

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